

# Evolution of the Carter constant for a resonant inspiral into a Kerr black hole: I. The scalar case

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We discuss the inspiral of a small body around a Kerr black hole. When the time scale of the radiation reaction is sufficiently longer than its orbital period, the leading order orbital evolution is described only by the knowledge of the averaged evolution of the constants of motion, *i.e.*, the energy, azimuthal angular momentum and the Carter constant. Although there is no conserved current composed of the perturbation field corresponding to the Carter constant, it has been shown that the averaged rate of change of the Carter constant can be given by a simple formula, when there exists a simultaneous turning point of the radial and polar oscillations. However, an inspiralling orbit may cross a “resonance” point, where the frequencies of the radial and polar orbital oscillations are in a rational ratio. At the resonance point, one cannot find a simultaneous turning point in general. Hence, even for the averaged rate of change of the Carter constant, a direct computation of the “self-force”, which is still challenging especially in the case of the Kerr background, seems to be necessary. In this paper, we present a method of computing the averaged rate of change of the Carter constant in a relatively simple manner at the “resonance” point.

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## I. INTRODUCTION

The study of the two-body problem in general relativity has been attracting much attention in recent years to obtain a better understanding of the gravitational wave emission from a binary system. If one body is much less massive than the other, *i.e.*, when we consider extreme mass ratio inspirals of a compact object into a super massive black hole, the compact object can be treated as a point particle and the two-body problem is handled within the self-force program (See reviews [1, 2] and references therein.)

When a nonspinning particle with mass  $\mu$  is moving along a bound orbit on a Kerr black hole with mass  $M (\gg \mu)$  and the spin parameter  $a$ , the long-term evolution of the orbit due to the radiation reaction of the gravitational wave emission is dictated by the long-time averaged rates of change of three constants of motion: the energy  $E$ , the azimuthal angular momentum  $L$  and the Carter constant  $Q$  [3] since the characteristic time scale of the secular orbital evolution is sufficiently longer than the orbital period. The orbital phase errors originating from the other effects scales as  $O((M/\mu)^0)$ . This particular approximation of the orbital evolution is referred to as the adiabatic approximation [4–8].

On average, the losses of the energy and the azimuthal angular momentum of the particle are to be balanced with those carried by gravitational waves to the infinity or into the Kerr horizon thanks to their global conservation laws. This is nothing but the well-established balance argument. Although we do not have any balance argument for the Carter constant, based on Mino’s argument [9], Sago *et al.* [10, 11] (see also Drasco *et al.* [12] for the scalar case) have succeeded in deriving a formula for the averaged rate of change of the Carter constant  $\langle dQ/dt \rangle$  similar to the one obtained by the balance argument for the energy and azimuthal angular momentum.

However, the above mentioned work assumes that the particle’s radial and polar orbital frequencies,  $\Omega_r$  and  $\Omega_\theta$ , are in an irrational ratio. When  $\Omega_r$  and  $\Omega_\theta$  cross a resonance point, at which  $\Omega_r/\Omega_\theta = j_r/j_\theta$  holds with coprime integers  $j_r$  and  $j_\theta$ , the above formula is not correct, and the phase error due to the ignorance of the correct formula can be as large as  $O((M/\mu)^{1/2})$ . This was first pointed out by Mino [4] (See also Ref. [5]). Recently, Flanagan and Hinderer [13] investigated this issue quantitatively within the post-Newtonian approximation. For the inspiral in the extreme mass ratio regime  $M \gg \mu$ , this phase error is much larger than the one from the first order conservative self-forces or the long-time averaged second order dissipative self-forces,

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which scale as  $O((M/\mu)^0)$  [5–7, 14–16]. For this reason, it is very demanded to derive a formula for  $\langle dQ/dt \rangle$  in the adiabatic approximation for a resonance orbit.

The main differences from the off-resonance case in deriving  $\langle dQ/dt \rangle$  are in the following two points. First, the orbit is no longer characterized by the three constants of motion alone. Instead, as was pointed out by Flanagan *et al.* [17], the orbit depends on the relative offset phase specified by  $\Delta\lambda$ , the separation between the instances when the  $r$ - and  $\theta$ - oscillations reach their minima. The offset phase also slowly evolves compared to the orbital period near resonance points. Thus, the resonance orbit must be characterized by four parameters:  $\{E, L_z, Q, \Delta\lambda\}$ . Second, we cannot use the Mino’s argument [4], which proves that  $\langle dQ/dt \rangle$  can be evaluated by using the radiative field defined by “the half-retarded minus half-advanced field”, which is by definition regular at the location of the particle. The key ingredient of the Mino’s argument is the invariance of the geodesics under the transformation  $(t, r, \theta, \phi) \rightarrow (-t, r, \theta, -\phi)$  that is assured if and only if the  $r$ - and  $\theta$ - components of the body’s four velocity simultaneously vanish at some time, *i.e.*, when we can choose  $\Delta\lambda = 0$ . Clearly, this condition is not satisfied for generic resonance geodesics.

Our final goal is to obtain the evolution of a resonant inspiral with the phase accuracy better than unity. Toward this ambitious goal, we here begin with deriving a practical formula of  $\langle dQ/dt \rangle$  for resonant inspirals. In the resonance case, we find that  $\langle dQ/dt \rangle$  acquires both the contributions from the radiative and symmetric fields, the latter defined by “the half-retarded *plus* half-advanced field”. In general, the symmetric field diverges at the particle’s location, thus requires some regularization. In the later section, we will propose a new regularization scheme for the symmetric portion of  $\langle dQ/dt \rangle$ . The key point of our regularization method is that it is compatible with the Teukolsky formalism [18]. This makes the calculation much simpler than the self-force calculation, which is already numerically challenging for a particle orbiting the Schwarzschild black hole (see Ref. [1] for recent progress in this topic).

In this paper, to avoid complications, we focus on a point scalar particle coupled to a massless field with the mass  $\mu$  and the charge  $q$ . Although there seems no essential obstacle in extending our scalar formalism to the gravitational case in principle, the latter is more involved since it requires the reconstruction of the metric perturbations, which is a little complicated in the Kerr case [19, 20].

The remainder of this manuscript is organized as follows. In Sec. II, we begin with describing the model of a point scalar particle coupled to a massless scalar field. Then, we review the generic bounded geodesic motion around a Kerr black hole, particularly focusing on the resonance case. The retarded solution of the scalar field equation expressed in the form of the mode decomposition is also summarized there. Then, we derive a general formula for the long-time averaged rate of change of the Carter constant in Sec. III. We separate the formula into two pieces, the radiative and the symmetric parts that originate from the radiative and symmetric fields, respectively. The result for the radiative part is shown in Sec. IIIB, and the result for the symmetric part is presented in Sec. IIIC, where we also discuss the mode sum regularization of the symmetric part since it diverges at the particle’s location. Finally, we summarize this paper in Sec. IV. To clarify the impact of our work, we also present there how the long-term evolution of the resonance orbit is described in the adiabatic approximation. In Appendix A, we discuss the possibility that the resonance effect on the phase error might be larger than  $O((\mu/M)^{-1/2})$ . Some detailed derivations related to the regularized symmetric part are summarized in Appendix B.

Throughout this paper, we use geometrical units  $G = c = 1$  and the sign convention of the metric is  $(-, +, +, +)$ . We basically follow the notation used by Drasco *et al.* [12].

## II. PRELIMINARIES: THE MOTION OF A POINT SCALAR PARTICLE

We consider a point scalar particle with the bare rest mass  $\mu = 1$ <sup>1</sup> and the scalar charge  $q$ , moving in the Kerr spacetime. The Kerr metric in the Boyer-Lindquist coordinates is given by:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2, \quad (1)$$

where  $M$  and  $a$  are the mass and the Kerr parameter, respectively, and

$$\Sigma := r^2 + a^2 \cos^2\theta, \quad \Delta := r^2 - 2Mr + a^2. \quad (2)$$

There are two Killing vectors associated with the stationarity and axisymmetry of the Kerr spacetime:

$$\xi_{(t)}^\mu := (\partial_t)^\mu = (1, 0, 0, 0), \quad \xi_{(\phi)}^\mu := (\partial_\phi)^\mu = (0, 0, 0, 1).$$

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<sup>1</sup> Strictly speaking,  $\mu$  is not constant because of the self-force effect. However, the correction due to this time dependence is  $O((M/\mu)^0)$  in the orbital phase, *i.e.* suppressed as well as the conservative self-force correction. Since we focus on the correction of  $O((M/\mu)^{1/2})$ , we neglect the change of the mass in this paper.

The Kerr spacetime also has an irreducible Killing tensor given by

$$K_{\mu\nu} := 2\Sigma l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu} = 2\Sigma m_{(\mu} \bar{m}_{\nu)} - a^2 \cos^2 \theta g_{\mu\nu}, \quad (3)$$

with the Kinnersley null tetrad defined by  $l^\mu := (r^2 + a^2, \Delta, 0, a)/\Delta$ ,  $n^\mu := (r^2 + a^2, -\Delta, 0, a)/(2\Sigma)$  and  $m^\mu := (ia \sin \theta, 0, 1, i/\sin \theta)/(\sqrt{2}(r + ia \cos \theta))$  where  $l^\mu n_\mu = -1$  and  $m^\mu \bar{m}_\mu = 1$  while all the other inner products vanish.

#### A. Bounded geodesic motion around a Kerr black hole in resonance

Neglecting the effect of the self-interaction, a particle can be regarded as a test particle that moves along a geodesic on the background spacetime,  $z^\mu(\tau) := (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$  parametrized by the proper time  $\tau$ . There are three integrals of motion owing to the symmetries of the Kerr spacetime: the energy  $E$ , the azimuthal angular momentum  $L$  and the Carter constant  $Q$  [3], respectively expressed as

$$\begin{aligned} E &:= -u^\alpha \xi_\alpha^{(t)} = \left(1 - \frac{2Mr}{\Sigma}\right) u^t + \frac{2Mar \sin^2 \theta}{\Sigma} u^\phi, \\ L &:= u^\alpha \xi_\alpha^{(\phi)} = -\frac{2Mar \sin^2 \theta}{\Sigma} u^t + \frac{(r^2 + a^2) - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta u^\phi, \\ Q &:= u^\alpha u^\beta K_{\alpha\beta} = \left(\frac{L}{\sin \theta} - a \sin \theta E\right)^2 + a^2 \cos^2 \theta + \Sigma^2 (u^\theta)^2, \end{aligned} \quad (4)$$

where  $u^\alpha := dz^\alpha/d\tau$  is the particle's four velocity. (Note that  $E$  and  $L$  are  $O(\mu)$ , and  $Q$  is  $O(\mu^2)$  if we recover  $\mu$ .)

For later analysis, it is convenient to parametrize the particle's orbit as  $z(\lambda) := (t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$  with the ‘‘Mino time’’  $\lambda$ , which is related to  $\tau$  as  $d\lambda := d\tau/\Sigma$  [4]. Using  $\lambda$ , the equations of motion are written as

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r), \quad \left(\frac{d\cos \theta}{d\lambda}\right)^2 = \Theta(\cos \theta), \quad (5)$$

$$\frac{dt}{d\lambda} = V_{t\theta}(\theta) + V_{tr}(r), \quad \frac{d\phi}{d\lambda} = V_{\phi\theta}(\theta) + V_{\phi r}(r), \quad (6)$$

where

$$\begin{aligned} R(r) &:= P(r)^2 - \Delta(r^2 + Q), \\ P(r) &:= E(r^2 + a^2) - aL, \\ \Theta(\cos \theta) &:= C - \{C + a^2(1 - E^2) + L^2\} \cos^2 \theta + a^2(1 - E^2) \cos^4 \theta, \\ V_{t\theta}(\theta) &:= -a(aE \sin^2 \theta - L), \quad V_{tr}(r) := \frac{r^2 + a^2}{\Delta} P(r), \\ V_{\phi\theta}(\theta) &:= -aE + \frac{L}{\sin^2 \theta}, \quad V_{\phi r}(r) := \frac{a}{\Delta} P(r), \end{aligned}$$

with  $C := Q - (aE - L)^2$ . Here it should be noted that the equations of  $r$  and  $\theta$  are completely decoupled, and that those of  $t$  and  $\phi$  are divided into  $r$  and  $\theta$  dependent parts.

First, it follows from Eqs. (5) that the  $r$ - and  $\theta$ - oscillations are independently periodic for a bound orbit:

$$r(\lambda) = r(\lambda + n\Lambda_r), \quad \theta(\lambda) = \theta(\lambda + k\Lambda_\theta), \quad (7)$$

where  $n$  and  $k$  are integers and the periods  $\Lambda_r$  and  $\Lambda_\theta$  with respect to the Mino time  $\lambda$  are given by

$$\Lambda_r := 2 \int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{R(r)}}, \quad \Lambda_\theta := 4 \int_0^{\cos \theta_{\min}} \frac{d(\cos \theta)}{\sqrt{\Theta(\theta)}} \quad (8)$$

with the turning points  $r_{\min}$ ,  $r_{\max}$  and  $\theta_{\min} < \pi/2$ . For later convenience, we introduce  $\bar{r}$  and  $\bar{\theta}$  which denote the fiducial solutions of Eqs. (5) such that  $\bar{r}(n\Lambda_r) = r_{\min}$  and  $\bar{\theta}(k\Lambda_\theta) = \theta_{\min}$ .

The precise meaning of the resonance is that the orbital frequencies

$$\Upsilon_r := \frac{2\pi}{\Lambda_r}, \quad \Upsilon_\theta := \frac{2\pi}{\Lambda_\theta}$$

are related to each other as

$$\frac{\Upsilon_r}{j_r} = \frac{\Upsilon_\theta}{j_\theta}, \quad (9)$$

where  $j_r$  and  $j_\theta$  are coprime integers. We here define the resonance frequency,  $\Upsilon$ , and the corresponding period,  $\Lambda$ , as

$$\Upsilon := \frac{\Upsilon_r}{j_r} = \frac{\Upsilon_\theta}{j_\theta}, \quad \Lambda := \frac{2\pi}{\Upsilon}.$$

In the resonance case, in general, there is a difference between the Mino times of reaching the minima of  $r$ - and  $\theta$ - oscillations. We call this difference “offset phase” and denote it by  $\Delta\lambda$ . Note that  $\Delta\lambda$  can be shifted by

$$\Lambda' := \frac{\Lambda}{j_r j_\theta}$$

depending on which minima we focus on. Therefore, we choose  $\Delta\lambda$  such that  $|\Delta\lambda| \leq \Lambda'/2$ . Setting the origin of the  $\lambda$  coordinate to satisfy  $\theta(\lambda) = \bar{\theta}(\lambda)$ , the orbit is written as

$$r(\lambda) = \bar{r}(\lambda - \Delta\lambda), \quad \theta(\lambda) = \bar{\theta}(\lambda) \quad (10)$$

without loss of generality.

Second, the motion in the  $t$ -direction is also separated into  $r$ - and  $\theta$ -dependent parts. We introduce the averaged values of  $V_{tr}(r)$  and  $V_{t\theta}(\theta)$  defined by

$$\langle V_{tr} \rangle := \frac{1}{\Lambda_r} \int_0^{\Lambda_r} d\lambda' V_{tr}[\bar{r}(\lambda')], \quad \langle V_{t\theta} \rangle := \frac{1}{\Lambda_\theta} \int_0^{\Lambda_\theta} d\lambda' V_{t\theta}[\bar{\theta}(\lambda')]. \quad (11)$$

In integrating the  $t$ -component of the equation of motion, the choice of the origin of the  $t$ -coordinate can be changed freely because of the time-translation symmetry of the Kerr spacetime. Then, without loss of generality,  $t(\lambda)$  is given by [12]

$$t(\lambda) = \Upsilon_t \lambda + \Delta t_r(\lambda - \Delta\lambda) + \Delta t_\theta(\lambda), \quad (12)$$

where  $\Upsilon_t := \langle V_{tr} \rangle + \langle V_{t\theta} \rangle$ , and

$$\Delta t_r(\lambda) := \int_0^\lambda d\lambda' \{V_{tr}[\bar{r}(\lambda')] - \langle V_{tr} \rangle\}, \quad \Delta t_\theta(\lambda) := \int_0^\lambda d\lambda' \{V_{t\theta}[\bar{\theta}(\lambda')] - \langle V_{t\theta} \rangle\}. \quad (13)$$

The function  $\phi(\lambda)$  is deduced in the same way as  $t(\lambda)$ . Again considering the axial symmetry of the Kerr spacetime, we have

$$\phi(\lambda) = \Upsilon_\phi \lambda + \Delta\phi_r(\lambda - \Delta\lambda) + \Delta\phi_\theta(\lambda), \quad (14)$$

where we define  $\Upsilon_\phi := \langle V_{\phi r} \rangle + \langle V_{\phi\theta} \rangle$  and

$$\Delta\phi_r(\lambda) := \int_0^\lambda d\lambda' \{V_{\phi r}[\bar{r}(\lambda')] - \langle V_{\phi r} \rangle\}, \quad \Delta\phi_\theta(\lambda) := \int_0^\lambda d\lambda' \{V_{\phi\theta}[\bar{\theta}(\lambda')] - \langle V_{\phi\theta} \rangle\} \quad (15)$$

with

$$\langle V_{\phi r} \rangle := \frac{1}{\Lambda_r} \int_0^{\Lambda_r} d\lambda' V_{\phi r}[\bar{r}(\lambda')], \quad \langle V_{\phi\theta} \rangle := \frac{1}{\Lambda_\theta} \int_0^{\Lambda_\theta} d\lambda' V_{\phi\theta}[\bar{\theta}(\lambda')]. \quad (16)$$

From the above discussion, the orbit is expressed as

$$z(\lambda) = \bar{z}(\lambda - \Delta\lambda, \lambda) \quad (17)$$

with the two variable function

$$\bar{z}(\lambda_r, \lambda_\theta) = [\Upsilon_t \lambda_\theta, 0, 0, \Upsilon_\phi \lambda_\theta] + \Delta \bar{z}(\lambda_r, \lambda_\theta), \quad (18)$$

where the oscillating part of the orbit is denoted by

$$\Delta \bar{z}(\lambda_r, \lambda_\theta) = [\Delta t_r(\lambda_r) + \Delta t_\theta(\lambda_\theta), \bar{r}(\lambda_r), \bar{\theta}(\lambda_\theta), \Delta\phi_r(\lambda_r) + \Delta\phi_\theta(\lambda_\theta)]. \quad (19)$$

### B. The retarded solution of the scalar field equation

The equation of the scalar field induced by a charged particle is given by

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi(x) = \mathcal{T}(x), \quad \mathcal{T}(x) := -q \int_{-\infty}^{+\infty} d\tau \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g(z(\tau))}}, \quad (20)$$

where  $\nabla_\mu$  and  $g$  show the covariant differentiation and the determinant for the background Kerr metric,  $g_{\mu\nu}$ , in Eq. (1), respectively. (See Ref. [12] for further details). Assuming that there is no incoming scalar wave, we take the retarded solution,  $\Phi^{(\text{ret})}$ . To obtain the solution, we here construct the retarded Green function in terms of the mode functions of the field equation.

From the fact that the scalar field equation is separable in the Kerr spacetime [18, 21], we can write the mode functions in the form of

$$\pi_{\omega lm}^{\flat}(t, r, \theta, \phi) := \frac{2}{\sqrt{r^2 + a^2}} e^{-i\omega t} u_{\omega lm}^{\flat}(r^*) S_{\omega lm}(\theta, \phi), \quad (21)$$

where  $r^*$  is the tortoise coordinate defined by  $dr^* := ((r^2 + a^2)/\Delta)dr$ , and  $S_{\omega lm}(\theta, \phi) := \Theta_{\omega lm}(\theta)e^{im\phi}$  is the spheroidal harmonics normalized as

$$\int d\theta d\phi \sin\theta S_{\omega lm}(\theta, \phi)^* S_{\omega l' m'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \quad (22)$$

The superscript  $\flat$  represents one of the four distinct boundary conditions: “up”, “down”, “in” and “out”. The corresponding radial functions are respectively defined by [12, 22]

$$\begin{aligned} u_{\omega lm}^{\text{in}}(r^*) &= \alpha_{\omega lm} \begin{cases} \tau_{\omega lm} |p_{m\omega}|^{-1/2} e^{-ip_{m\omega} r^*}, & (r^* \rightarrow -\infty), \\ |\omega|^{-1/2} [e^{-i\omega r^*} + \sigma_{\omega lm} e^{i\omega r^*}], & (r^* \rightarrow \infty), \end{cases} \\ u_{\omega lm}^{\text{up}}(r^*) &= \beta_{\omega lm} \begin{cases} |p_{m\omega}|^{-1/2} \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} [\mu_{\omega lm} e^{ip_{m\omega} r^*} + \nu_{\omega lm} e^{-ip_{m\omega} r^*}], & (r^* \rightarrow -\infty), \\ |\omega|^{-1/2} e^{i\omega r^*}, & (r^* \rightarrow \infty), \end{cases} \\ u_{\omega lm}^{\text{out}}(r^*) &= u_{\omega lm}^{\text{in}*}(r^*), \\ u_{\omega lm}^{\text{down}}(r^*) &= u_{\omega lm}^{\text{up}*}(r^*), \end{aligned} \quad (23)$$

with  $p_{m\omega} := \omega - ma/(2Mr_+)$  and  $r_+ := M + \sqrt{M^2 - a^2}$ . The coefficients  $\alpha_{\omega lm}$  and  $\beta_{\omega lm}$  are normalization constants.  $\tau_{\omega lm}$  and  $\sigma_{\omega lm}$  are, respectively, complex transmission and reflection coefficients. Both the complex coefficients  $\mu_{\omega lm}$  and  $\nu_{\omega lm}$  are also defined by Eq. (23). In this work, we choose the normalizations of the “in” and “up” solutions as

$$\alpha_{\omega lm}^* = \alpha_{-\omega l - m}, \quad \beta_{\omega lm}^* = \beta_{-\omega l - m}, \quad (24)$$

so that the invariance of the radial equation under the transformation  $(\omega, m) \rightarrow (-\omega, -m)$  implies the identity

$$u_{-\omega l - m}^{\flat}(r^*) = u_{\omega lm}^{\flat*}(r^*). \quad (25)$$

From the identity (25), we find

$$\tau_{\omega lm}^* = \tau_{-\omega l - m}, \quad \sigma_{\omega lm}^* = \sigma_{-\omega l - m}, \quad \mu_{\omega lm}^* = \mu_{-\omega l - m}, \quad \nu_{\omega lm}^* = \nu_{-\omega l - m}. \quad (26)$$

It should be also noted that we choose the phase of the normalization of the angular function  $\Theta_{\omega lm}(\theta)$  so that

$$\Theta_{-\omega l - m}^*(\theta) = (-1)^m \Theta_{\omega lm}(\theta). \quad (27)$$

The retarded Green function for the scalar field equation (20) is expressed as the factorized form in terms of the mode functions [12]:

$$\begin{aligned} G^{(\text{ret})}(x, x') &= \frac{1}{16\pi i} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l \frac{\omega}{|\omega|} \frac{1}{\alpha_{\omega lm} \beta_{\omega lm}} [\pi_{\omega lm}^{\text{up}}(x) \pi_{\omega lm}^{\text{out}*}(x') H(r - r') + \pi_{\omega lm}^{\text{in}}(x) \pi_{\omega lm}^{\text{down}*}(x') H(r' - r)] \\ &= \frac{1}{16\pi i} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l \frac{\omega}{|\omega|} \frac{1}{\alpha_{\omega lm} \beta_{\omega lm}} e^{-i\omega(t-t') + im(\phi-\phi')} \Theta_{\omega lm}(\theta) \Theta_{\omega lm}^*(\theta') \end{aligned}$$

$$\times \left[ R_{\omega lm}^{\text{up}}(r) R_{\omega lm}^{\text{out}*}(r') H(r - r') + R_{\omega lm}^{\text{out}*}(r) R_{\omega lm}^{\text{up}}(r') H(r' - r) \right] \quad (28)$$

with the Heaviside function  $H(x) := \int_{-\infty}^x \delta(y) dy$ , and  $R_{\omega lm}^b(r) \equiv 2u_{\omega lm}^b / \sqrt{r^2 + a^2}$ . In showing the second equality of Eq. (28), we used the relations between the radial functions  $u_{\omega lm}^b(r^*)$  in Eq. (23). This Green function leads the retarded solution as

$$\Phi^{(\text{ret})}(x) = \int d^4 x' \sqrt{-g(x')} G^{(\text{ret})}(x, x') \mathcal{T}(x') = -q \int d\tau G^{(\text{ret})}[x, z(\tau)]. \quad (29)$$

It is helpful for the later analysis to decompose the retarded field into a linear combination of the radiative (anti-symmetric) and symmetric fields as

$$\Phi^{(\text{ret})}(x) := \Phi^{(\text{rad})}(x) + \Phi^{(\text{sym})}(x), \quad (30)$$

where

$$\Phi^{(\text{rad})}(x) = -q \int d\tau G^{(\text{rad})}[x, z(\tau)], \quad \Phi^{(\text{sym})}(x) = -q \int d\tau G^{(\text{sym})}[x, z(\tau)] \quad (31)$$

with

$$G^{(\text{rad})}(x, x') = \frac{1}{2} \left[ G^{(\text{ret})}(x, x') - G^{(\text{adv})}(x, x') \right], \quad (32)$$

$$G^{(\text{sym})}(x, x') = \frac{1}{2} \left[ G^{(\text{ret})}(x, x') + G^{(\text{adv})}(x, x') \right]. \quad (33)$$

Here the advanced Green function,  $G^{(\text{adv})}(x, x')$ , is derived from the retarded Green function as

$$G^{(\text{adv})}(x, x') = G^{(\text{ret})}(x', x). \quad (34)$$

The radiative Green function can be rewritten as [12]

$$G^{(\text{rad})}(x, x') = \frac{1}{32\pi i} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l \frac{\omega}{|\omega|} \left[ \frac{1}{|\alpha_{\omega lm}|^2} \pi_{\omega lm}^{\text{out}}(x) \pi_{\omega lm}^{\text{out}*}(x') + \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} \frac{|\tau_{\omega lm}|^2}{|\beta_{\omega lm}|^2} \pi_{\omega lm}^{\text{down}}(x) \pi_{\omega lm}^{\text{down}*}(x') \right], \quad (35)$$

and the Heaviside functions are eliminated. On the other hand, we rewrite the symmetric Green function in the following way. Writing  $G^{(\text{adv})}(x, x')$  in terms of the mode functions in a similar manner to Eq. (28), replacing  $(\omega, m)$  to  $(-\omega, -m)$ , and rearranging it with Eqs. (24), (25), (26), and (27), we find

$$G^{(\text{adv})}(x, x') = \frac{-1}{16\pi i} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l \frac{\omega}{|\omega|} \frac{1}{\alpha_{\omega lm}^* \beta_{\omega lm}^*} e^{-i\omega(t-t') + im(\phi-\phi')} \Theta_{\omega lm}(\theta) \Theta_{\omega lm}^*(\theta') \\ \times \left[ R_{\omega lm}^{\text{up}*}(r) R_{\omega lm}^{\text{out}}(r') H(r - r') + R_{\omega lm}^{\text{out}}(r) R_{\omega lm}^{\text{up}*}(r') H(r' - r) \right]. \quad (36)$$

Substituting Eqs. (28) and (36) into Eq. (33), we find that the symmetric Green function is rewritten as

$$G^{(\text{sym})}(x, x') = \frac{1}{16\pi} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l g_{\omega lm}^{(\text{sym})}(x, x'), \quad (37)$$

where

$$g_{\omega lm}^{(\text{sym})}(x, x') = e^{-i\omega(t-t') + im(\phi-\phi')} \Theta_{\omega lm}(\theta) \Theta_{\omega lm}^*(\theta') \\ \times \Im \left[ \frac{\omega}{|\omega|} \frac{1}{\alpha_{\omega lm} \beta_{\omega lm}} \left[ R_{\omega lm}^{\text{up}}(r) R_{\omega lm}^{\text{out}*}(r') H(r - r') + R_{\omega lm}^{\text{out}*}(r) R_{\omega lm}^{\text{up}}(r') H(r' - r) \right] \right], \quad (38)$$

and  $\Im[X]$  represents the imaginary part of  $X$ .

### III. ADIABATIC EVOLUTION OF THE CARTER CONSTANT AT RESONANCE

We denote the three constants of motion as  $I^i := \{E, L, Q\}$ . The time averaged rates of change of  $I^i$  can be reexpressed as

$$\left\langle \frac{dI^i}{dt} \right\rangle_t = \frac{1}{\Upsilon_t} \left\langle \frac{dI^i}{d\lambda} \right\rangle_\lambda, \quad (39)$$

with the definition of the  $\lambda$  average,

$$\langle F(\lambda) \rangle_\lambda := \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} d\lambda' F(\lambda'). \quad (40)$$

One key property of the adiabatic approximation developed in Ref. [4] for off-resonance orbits is that  $\langle dI^i/d\lambda \rangle_\lambda$  can be evaluated by using the radiative solution of the field equation (20) as

$$\left\langle \frac{dI^i}{d\lambda} \right\rangle_\lambda = \left\langle \frac{dI^i}{du^\alpha} [\Sigma(x) f^\alpha[\Phi^{(\text{rad})}(x)]]_{x=z(\lambda)} \right\rangle_\lambda, \quad (41)$$

where  $f^\alpha$  is a differential operator defined by [23–25]

$$f^\alpha[\Phi(x)] := q(g^{\alpha\beta} + u^\alpha u^\beta) \nabla_\beta \Phi(x). \quad (42)$$

Since the radiative field has no divergent behaviour at the location of the particle, this relation gives  $\langle dI^i/d\lambda \rangle_\lambda$  without any regularization procedure. The final expressions for  $\langle dI^i/d\lambda \rangle_\lambda$  are given in terms of the asymptotic amplitudes of the scalar field at infinity and on the horizon [12].

The derivation of Eq.(41) relies on the assumption that there exists a simultaneous turning point of the  $r$ - and  $\theta$ - oscillations of the particle. In the case of a resonance orbit, however, the frequencies of the  $r$ - and  $\theta$ - oscillations are in a rational ratio, and hence one cannot find a simultaneous turning point in general. This means that the orbit depends on the non-vanishing offset phase,  $\Delta\lambda$ . Therefore, the expressions for  $\langle dI^i/dt \rangle_t$  should be derived without using the relation (41).

As for  $\langle dE/dt \rangle_t$  and  $\langle dL/dt \rangle_t$ , the balance argument can be used to relate them to the fluxes of the energy and angular momentum to the infinity or into the black hole horizon. Therefore, one can conclude that the expressions for  $\langle dE/dt \rangle_t$  and  $\langle dL/dt \rangle_t$  in the resonance case are the same as those in the non-resonance case [17]. On the other hand, there is no known flux composed of the perturbation field corresponding to the Carter constant, and hence we cannot use the balance argument to derive the averaged rate of change,  $\langle dQ/dt \rangle_t$ . We need to reformulate the calculation of  $\langle dQ/dt \rangle_t$  without using the relation (41). In this section, we derive a formula for computing  $\langle dQ/dt \rangle_t$  in the resonance case, by starting from the basic formula of the self-force acting on a particle.

#### A. Formal reduction of $\langle dQ/dt \rangle_t$ for a resonance orbit

In contrast to  $\langle dE/dt \rangle_t$  and  $\langle dL/dt \rangle_t$ , we need to directly evaluate the time derivative of the Carter constant defined in Eq. (4) caused by the self-force acting on the particle, which is given by

$$\frac{dQ}{d\lambda} = 2\Sigma K_{\alpha\beta} u^\alpha a^\beta \quad (43)$$

with the aid of the Killing tensor equation  $\nabla_{(\gamma} K_{\alpha\beta)} = 0$ . Here  $a^\alpha$  is the four acceleration due to the scalar self-forces with the retarded scalar field given by [26]

$$a^\alpha[z(\tau)] := \lim_{x \rightarrow z(\tau)} f^\alpha[\Phi^{(R)}(x)], \quad (44)$$

where  $\Phi^{(R)}$  is the so-called  $R$ -part of the retarded solution, which satisfies the homogeneous equation of Eq. (20).  $\Phi^{(R)}$  is schematically expressed as

$$\begin{aligned} \Phi^{(R)}(x) &:= \Phi^{(\text{ret})}(x) - \Phi^{(S)}(x) \\ &= \Phi^{(\text{rad})}(x) + \left\{ \Phi^{(\text{sym})}(x) - \Phi^{(S)}(x) \right\}, \end{aligned} \quad (45)$$

where  $\Phi^{(S)}(x)$  is the  $S$ -part of the retarded field [26], which satisfies the same equation as the retarded (or symmetric) solution. It should be noted that, the difference between  $\Phi^{(\text{sym})}$  and  $\Phi^{(S)}$  is regular at the location of the particle while each of them is singular.



Now plugging Eq. (44) into Eq. (43), the time derivative of the Carter constant is rewritten in terms of the  $R$ -part of the retarded field  $\Phi^{(R)}(x)$ :

$$\frac{dQ}{d\lambda} = 2Qq \frac{d\Phi^{(R)}}{d\lambda} + 2q\Sigma K^{\alpha\beta} u_\alpha \nabla_\beta \Phi^{(R)}. \quad (46)$$

Since the first term is a total derivative up to the leading order in  $q$ , it does not contribute after taking a long-time average. Hence, we omit the first term in Eq. (46). Substituting the explicit expression for  $K_{\alpha\beta}$  into Eq. (46), and dropping the first term, we obtain

$$\left\langle \frac{dQ}{dt} \right\rangle_t = -\frac{2q}{\Upsilon_t} \left\langle \left\{ \hat{\mathcal{O}}_x \left[ \Sigma(x) \Phi^{(R)}(x) \right] \right\}_{x=z(\lambda)} \right\rangle_\lambda, \quad (47)$$

where  $\hat{\mathcal{O}}_x$  is a differential operator defined by

$$\hat{\mathcal{O}}_x := V_{tr}(r) \partial_t + V_{\phi r}(r) \partial_\phi + \frac{dr(\lambda)}{d\lambda} \partial_r. \quad (48)$$

Following the decomposition of the retarded field in Eq. (45), we also decompose  $\langle dQ/dt \rangle_t$  into two parts as

$$\left\langle \frac{dQ}{dt} \right\rangle_t = \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{rad})} + \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{sym}-S)} \quad (49)$$

with

$$\begin{aligned} \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{rad})} &= \frac{2q^2}{\Upsilon_t} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda \int_{-\infty}^{\infty} d\lambda' \left\{ \hat{\mathcal{O}}_x \left[ \Sigma(x) \Sigma(x') G^{(\text{rad})}(x, x') \right] \right\}_{x=z(\lambda), x'=z(\lambda')}, \\ \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{sym}-S)} &= \frac{2q^2}{\Upsilon_t} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda \int_{-\infty}^{\infty} d\lambda' \left\{ \hat{\mathcal{O}}_x \left[ \Sigma(x) \Sigma(x') \left( G^{(\text{sym})}(x, x') - G^{(S)}(x, x') \right) \right] \right\}_{x=z(\lambda), x'=z(\lambda')}, \end{aligned} \quad (50)$$

where  $G^{(S)}(x, x')$  is the  $S$ -part of the retarded Green function (see Eq. (81) below).

We reduce Eqs. (50) and (51) into more tractable expressions. For this purpose, we here consider a function of two variables,  $\tilde{G}(x, x') \equiv \Sigma(x) \Sigma(x') G(x, x')$ . Using the identity,

$$\frac{d}{d\lambda_r} \tilde{G}(\bar{z}(\lambda_r, \lambda_\theta), x') = \left[ \left( \frac{dr(\lambda_r)}{d\lambda_r} \partial_r + \frac{d\Delta t_r(\lambda_r)}{d\lambda_r} \partial_t + \frac{d\Delta\phi_r(\lambda_r)}{d\lambda_r} \partial_\phi \right) \tilde{G}(x, x') \right]_{x=\bar{z}(\lambda_r, \lambda_\theta)}, \quad (52)$$

we find

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda \left\{ \hat{\mathcal{O}}_x \left[ \tilde{G}(x, x') \right] \right\}_{x=z(\lambda)} \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda_\theta \int_{-\infty}^{+\infty} d\lambda_r \delta(\lambda_r - \lambda_\theta + \Delta\lambda) \left\{ \hat{\mathcal{O}}_x \left[ \tilde{G}(x, x') \right] \right\}_{x=\bar{z}(\lambda_r, \lambda_\theta)} \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda_\theta \int_{-\infty}^{+\infty} d\lambda_r \delta(\lambda_r - \lambda_\theta + \Delta\lambda) \left\{ \left[ (\langle V_{tr} \rangle \partial_t + \langle V_{\phi r} \rangle \partial_\phi) \tilde{G}(x, x') \right]_{x=\bar{z}(\lambda_r, \lambda_\theta)} + \frac{d}{d\lambda_r} \left[ \tilde{G}(\bar{z}(\lambda_r, \lambda_\theta), x') \right] \right\} \\ &= \left\langle \left[ (\langle V_{tr} \rangle \partial_t + \langle V_{\phi r} \rangle \partial_\phi) \tilde{G}(x, x') \right]_{x=z(\lambda)} \right\rangle_\lambda - \frac{d}{d(\Delta\lambda)} \langle \tilde{G}[z(\lambda), x'] \rangle_\lambda. \end{aligned} \quad (53)$$

Here we insert the  $\delta$ -function  $\delta(\lambda_r - \lambda_\theta + \Delta\lambda)$  on the second line to deal with the  $r$ - and  $\theta$ -oscillations separately, keeping the information of the offset phase. In the last equality, we performe integration by parts with respect to  $\lambda_r$  and replace  $d\delta(\lambda_r - \lambda_\theta + \Delta\lambda)/d\lambda_r$  with  $d\delta(\lambda_r - \lambda_\theta + \Delta\lambda)/d(\Delta\lambda)$ . The differentiation  $d/d(\Delta\lambda)$  can be extracted to the outside of the integral because the integrand depends on  $\Delta\lambda$  only through this  $\delta$ -function.

As for  $\langle dQ/dt \rangle_t^{(\text{rad})}$ , with the aid of Eq.(53), we arrive at the following expression:

$$\begin{aligned} \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{rad})} &= \frac{2q^2}{\Upsilon_t} \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] \left\langle \left[ \Sigma(x) \left( \langle V_{tr} \rangle \partial_t + \langle V_{\phi r} \rangle \partial_\phi \right) G^{(\text{rad})}(x, z(\lambda')) \right]_{x=z(\lambda)} \right\rangle_\lambda \\ &\quad - \frac{2q^2}{\Upsilon_t} \int_{-\infty}^{\infty} d\lambda' \left[ \Sigma(x') \frac{d}{d(\Delta\lambda)} \left\langle \Sigma[z(\lambda)] G^{(\text{rad})}(z(\lambda), x') \right\rangle_\lambda \right]_{x'=z(\lambda')}. \end{aligned} \quad (54)$$



Notice that the  $\Delta\lambda$ -differentiation in the second term operates only through  $z(\lambda)$ , not through  $z(\lambda')$ .

For  $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$ , we obtain a similar formula to the radiative part as

$$\begin{aligned} \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{sym}-S)} &= \frac{2q^2}{\Upsilon_t} \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] \left\langle \left[ \Sigma(x) \left( \langle V_{tr} \rangle \partial_t + \langle V_{\phi r} \rangle \partial_\phi \right) G^{(\text{sym}-S)}(x, z(\lambda')) \right]_{x=z(\lambda)} \right\rangle_\lambda \\ &\quad - \frac{2q^2}{\Upsilon_t} \int_{-\infty}^{\infty} d\lambda' \left[ \Sigma(x') \frac{d}{d(\Delta\lambda)} \left\langle \Sigma[z(\lambda)] G^{(\text{sym}-S)}(z(\lambda), x') \right\rangle_\lambda \right]_{x'=z(\lambda')}, \end{aligned} \quad (55)$$

where

$$G^{(\text{sym}-S)}(x, x') := G^{(\text{sym})}(x, x') - G^{(S)}(x, x').$$

Using the following properties of  $G^{(\text{sym}-S)}(x, x')$  (See Appendix Appendix B for details.),

$$G^{(\text{sym}-S)}(x, x') = G^{(\text{sym}-S)}(x', x), \quad (56)$$

$$G^{(\text{sym}-S)}(z(\lambda), z(\lambda')) = G^{(\text{sym}-S)}(z(\lambda + \Lambda), z(\lambda' + \Lambda)), \quad (57)$$

$$\frac{\partial}{\partial t} G^{(\text{sym}-S)}(x, x') = -\frac{\partial}{\partial t'} G^{(\text{sym}-S)}(x, x'), \quad \frac{\partial}{\partial \phi} G^{(\text{sym}-S)}(x, x') = -\frac{\partial}{\partial \phi'} G^{(\text{sym}-S)}(x, x'), \quad (58)$$

we find that the first term in Eq. (55) vanishes to obtain

$$\left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{sym}-S)} = -\frac{q^2}{\Upsilon_t} \frac{d}{d(\Delta\lambda)} \left[ \int_{-\infty}^{\infty} d\lambda' \left\langle \Sigma[z(\lambda)] \Sigma[z(\lambda')] G^{(\text{sym}-S)}(z(\lambda), z(\lambda')) \right\rangle_\lambda \right]. \quad (59)$$

In Eq. (59), the differentiation with respect to  $\Delta\lambda$  operates on both  $z(\lambda)$  and  $z(\lambda')$ , while, in the second term of Eq. (55), it operates only on  $z(\lambda)$ . Since the part in the square brackets in Eq. (59) is symmetric under the exchange of  $z(\lambda)$  and  $z(\lambda')$ , however, the contribution from the  $\Delta\lambda$ -differentiation through  $z(\lambda')$  is the same as that through  $z(\lambda)$ . This is the reason why the factor 2 in Eq. (55) disappears in Eq. (59). The detailed derivation is also presented in Appendix B. The result that the  $t$ - and  $\phi$ -derivative parts in Eq. (55) have no contribution can be expected from the fact that  $\langle dE/dt \rangle_t$  and  $\langle dL/dt \rangle_t$  have no contribution from the symmetric part of the Green function, which is shown by the balance argument of the corresponding conserved currents. We emphasize that, from the above expression,  $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$  will not vanish in general. This means that the secular change of the Carter constant is likely to be also induced by the contribution from the symmetric part for the resonance orbit.

It should be noted that we cannot divide the integration in Eq. (59) into the symmetric and  $S$ -parts because both  $G^{(\text{sym})}[z(\lambda), z(\lambda')]$  and  $G^{(S)}[z(\lambda), z(\lambda')]$  diverge at the coincidence limit,  $z(\lambda) \rightarrow z(\lambda')$ . In order to treat them separately, we here introduce the point splitting regularization by displacing the orbits  $z(\lambda)$  and  $z(\lambda')$  as

$$z^\mu(\lambda) \rightarrow z_+^\mu(\lambda) := z^\mu(\lambda) + \frac{\epsilon}{2} \xi^\mu, \quad z^\mu(\lambda') \rightarrow z_-^\mu(\lambda') := z^\mu(\lambda') - \frac{\epsilon}{2} \xi^\mu \quad (60)$$

with a small parameter  $\epsilon \ll 1$  and the Killing field

$$\xi^\mu(\zeta) := \cos \zeta \xi_{(t)}^\mu + (\Omega_\phi \cos \zeta - \Omega \sin \zeta) \xi_{(\phi)}^\mu \quad (61)$$

as the direction of displacement.  $\zeta$  is a parameter that specifies the choice of the Killing field,  $\Omega_\phi = \Upsilon_\phi / \Upsilon_t$  and  $\Omega = \Upsilon / \Upsilon_t$ . Under this point splitting regularization, we rewrite Eq. (59) as

$$\left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{sym}-S)} = -\lim_{\epsilon \rightarrow 0} \frac{q^2}{\Upsilon_t} \frac{d}{d(\Delta\lambda)} \left[ \Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta) - \Psi^{(S)}(\Delta\lambda; \epsilon, \zeta) \right]. \quad (62)$$

The potential functions,  $\Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta)$  and  $\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$ , are defined by

$$\Psi^{(X)}(\Delta\lambda; \epsilon, \zeta) = \int_{-\infty}^{+\infty} d\lambda' \left\langle \Sigma[z(\lambda)] \Sigma[z(\lambda')] G^{(X)}[z_+(\lambda), z_-(\lambda')] \right\rangle_\lambda, \quad (63)$$

where the superscript  $X$  is to be replaced to “sym” or “ $S$ ”. (Notice that, since  $\Sigma(x)$  is a function of  $r$  and  $\theta$ ,  $\Sigma[z_\pm(\lambda)] = \Sigma[z(\lambda)]$ .) It would be worth mentioning that the difference of the potential functions,  $\Psi^{(\text{sym}-S)}(\Delta\lambda; \epsilon, \zeta) := \Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta) - \Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$ , is even in the sign flip of  $\Delta\lambda$ . To show this, we replace  $\lambda$  and  $\lambda'$  with  $-\lambda$  and  $-\lambda'$ . Under this replacement, the offset phase  $\Delta\lambda$  (and the Killing vector  $\xi^\mu(\zeta)$ ) also flips the sign. Hence, we find  $\lim_{\epsilon \rightarrow 0} \Psi^{(\text{sym}-S)}(\Delta\lambda; \epsilon, \zeta) = \lim_{\epsilon \rightarrow 0} \Psi^{(\text{sym}-S)}(-\Delta\lambda; \epsilon, \zeta)$ . This symmetry guarantees  $\langle dQ/dt \rangle_t^{(\text{sym}-S)} = 0$  for  $\Delta\lambda = 0$  in general. This is consistent with the fact that the symmetric part does not contribute to  $\langle dQ/dt \rangle_t$  in the off-resonance case [9], in which we can set  $\Delta\lambda = 0$ .

Using the mode decomposed form of the radiative and symmetric Green functions given in Eqs. (35) and (37), the expressions in Eqs. (54) and (62) (with Eq. (63)) can be recast into more practical expressions. We will show them in the following subsections.

### B. Further reduction of $\langle dQ/dt \rangle_t^{(\text{rad})}$ : the radiative part

Flanagan *et al.* [17] have already written down a little more practical expression for the radiative part of the long-time averaged value of the rate of change of the Carter constant,  $\langle dQ/dt \rangle_t^{(\text{rad})}$ , in the gravitational case. Here, we briefly rederive the same result in the scalar case in a slightly different manner.

Our first task is to rewrite Eq. (54) in terms of the mode functions given in Eq. (21). With the aid of the radiative Green function (35) expressed as a sum of mode functions, we have

$$\begin{aligned} \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{rad})} &= \frac{q^2}{16\pi i \Upsilon_t} \int_{-\infty}^{+\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^l \frac{\omega}{|\omega|} \\ &\times \left[ \frac{1}{|\alpha_{\omega lm}|^2} \left\{ \int_{-\infty}^{+\infty} d\lambda' \tilde{\pi}_{\omega lm}^{\text{out}*}[z(\lambda')] \right\} \left( -i\omega \langle V_{tr} \rangle + im \langle V_{\phi r} \rangle - \frac{d}{d(\Delta\lambda)} \right) \langle \tilde{\pi}_{\omega lm}^{\text{out}}[z(\lambda)] \rangle_\lambda \right. \\ &\left. + \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} \frac{|\tau_{\omega lm}|^2}{|\beta_{\omega lm}|^2} \left\{ \int_{-\infty}^{+\infty} d\lambda' \tilde{\pi}_{\omega lm}^{\text{down}*}[z(\lambda')] \right\} \left( -i\omega \langle V_{tr} \rangle + im \langle V_{\phi r} \rangle - \frac{d}{d(\Delta\lambda)} \right) \langle \tilde{\pi}_{\omega lm}^{\text{down}}[z(\lambda)] \rangle_\lambda \right], \end{aligned} \quad (64)$$

where, for brevity, we introduced  $\tilde{\pi}_{\omega lm}^b(x) := \Sigma(x) \pi_{\omega lm}^b(x)$ , and also replaced the partial differentiations with respect to  $t$  and  $\phi$  with  $-i\omega$  and  $im$ , respectively. We can rewrite  $\tilde{\pi}_{\omega lm}^b[z(\lambda)]$  as

$$\tilde{\pi}_{\omega lm}^b[z(\lambda)] = \mathcal{J}_{\omega lm}^b(\lambda - \Delta\lambda, \lambda) e^{-i\lambda(\omega \Upsilon_t - m \Upsilon_\phi)} \quad (65)$$

with a function of two time variables,  $\lambda_r$  and  $\lambda_\theta$ :

$$\mathcal{J}_{\omega lm}^b(\lambda_r, \lambda_\theta) := \Sigma(x) \pi_{\omega lm}^{b*}(x) \Big|_{x=\Delta\bar{z}(\lambda_r, \lambda_\theta)}, \quad (66)$$

where  $\Delta\bar{z}(\lambda_r, \lambda_\theta)$  is the oscillating part of the orbit defined in Eq. (19). Since the function  $\mathcal{J}_{\omega lm}^b(\lambda_r, \lambda_\theta)$  is biperiodic with the frequencies  $\Upsilon_r$  and  $\Upsilon_\theta$  for  $\lambda_r$  and  $\lambda_\theta$ , respectively, it can be expanded in the form of a Fourier series as

$$\mathcal{J}_{\omega lm}^b(\lambda_r, \lambda_\theta) = \sum_{n_r=-\infty}^{\infty} \sum_{n_\theta=-\infty}^{\infty} \tilde{\mathcal{J}}_{lmn_r n_\theta}^b(\omega) e^{-i(n_r \Upsilon_r \lambda_r + n_\theta \Upsilon_\theta \lambda_\theta)}, \quad (67)$$

where

$$\tilde{\mathcal{J}}_{lmn_r n_\theta}^b(\omega) := \frac{1}{\Lambda_r} \int_0^{\Lambda_r} d\lambda_r \frac{1}{\Lambda_\theta} \int_0^{\Lambda_\theta} d\lambda_\theta \mathcal{J}_{\omega lm}^b(\lambda_r, \lambda_\theta) e^{i(n_r \Upsilon_r \lambda_r + n_\theta \Upsilon_\theta \lambda_\theta)}.$$

By using Eq. (65) with Eq. (67), we express the  $\lambda'$ -integrals in Eq. (64) as

$$\begin{aligned} \int_{-\infty}^{\infty} d\lambda' \tilde{\pi}_{\omega lm}^{b*}[z(\lambda')] &= \sum_{n_r=-\infty}^{\infty} \sum_{n_\theta=-\infty}^{\infty} \frac{2\pi}{\Upsilon_t} \tilde{\mathcal{J}}_{lmn_r n_\theta}^b(\omega) e^{in_r \Upsilon_r \Delta\lambda} \delta(\omega - \omega_{mn_r n_\theta}) \\ &= \sum_{N=-\infty}^{\infty} Z_{lmN}^b(\Delta\lambda) \delta(\omega - \omega_{mN}), \end{aligned} \quad (68)$$

where we introduced

$$\omega_{mn_r n_\theta} := \Upsilon_t^{-1} (m \Upsilon_\phi + n_r \Upsilon_r + n_\theta \Upsilon_\theta) = m \Omega_\phi + (j_r n_r + j_\theta n_\theta) \Omega, \quad (69)$$

$$\omega_{mN} := m \Omega_\phi + N \Omega, \quad (70)$$

and

$$Z_{lmN}^b(\Delta\lambda) := \frac{2\pi}{\Upsilon_t} \sum_{(n_r, n_\theta) \in \hat{N}} \tilde{\mathcal{J}}_{lmn_r n_\theta}^b(\omega_{mn_r n_\theta}) e^{in_r \Upsilon_r \Delta\lambda}, \quad (71)$$

with

$$\hat{N} := \{(n_r, n_\theta) | j_r n_r + j_\theta n_\theta = N\}. \quad (72)$$

Here it should be noted that  $N$  denotes an integer, and that we rearranged the order of summation over  $(n_r, n_\theta)$  in the second equality in Eq. (68).

Owing to the delta function in Eq. (68), we can replace the Fourier integral in Eq. (64) with a discrete Fourier series. Hence, we can also replace  $\omega$  in  $\langle \tilde{\pi}_{\omega_{mN}lm}^b[z(\lambda)] \rangle_\lambda$  with  $\omega_{mN}$  and then

$$\begin{aligned} \langle \tilde{\pi}_{\omega_{mN}lm}^b[z(\lambda)] \rangle_\lambda &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda \sum_{n_r, n_\theta} \tilde{g}_{lmn_r n_\theta}^{b*}(\omega_{mN}) e^{i\lambda(n_r \Upsilon_r + n_\theta \Upsilon_\theta - N \Upsilon) - i n_r \Upsilon_r \Delta \lambda} \\ &= \sum_{n_r, n_\theta} \tilde{g}_{lmn_r n_\theta}^{b*}(\omega_{mN}) e^{-i n_r \Upsilon_r \Delta \lambda} \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} d\lambda e^{i\lambda \Upsilon(n_r j_r + n_\theta j_\theta - N)} \\ &= \frac{\Upsilon_t}{2\pi} Z_{lmN}^{b*}(\Delta \lambda). \end{aligned} \quad (73)$$

In the second equality, we used the periodicity of the integrand with the period  $\Lambda$ .

Finally, using Eqs. (68) and (73), we can reduce Eq. (64) to

$$\begin{aligned} \left\langle \frac{dQ}{dt} \right\rangle_t^{(\text{rad})} &= -\frac{q^2}{32\pi^2} \sum_{\#} \frac{\omega_{mN}}{|\omega_{mN}|} \left[ (\omega_{mN} \langle V_{tr} \rangle - m \langle V_{\phi r} \rangle) \left\{ \frac{|Z_{lmN}^{\text{out}}(\Delta \lambda)|^2}{|\alpha_{\omega_{mN}lm}|^2} + \frac{\omega_{mN} p_{m\omega_{mN}}}{|\omega_{mN} p_{m\omega_{mN}}|} \frac{|\tau_{\omega_{mN}lm}|^2}{|\beta_{\omega_{mN}lm}|^2} |Z_{lmN}^{\text{down}}(\Delta \lambda)|^2 \right\} \right. \\ &\quad \left. - \Upsilon_r \left\{ \frac{1}{|\alpha_{\omega_{mN}lm}|^2} Z_{lmN}^{\text{out}}(\Delta \lambda) y_{lmN}^{\text{out}*}(\Delta \lambda) + \frac{\omega_{mN} p_{m\omega_{mN}}}{|\omega_{mN} p_{m\omega_{mN}}|} \frac{|\tau_{\omega_{mN}lm}|^2}{|\beta_{\omega_{mN}lm}|^2} Z_{lmN}^{\text{down}}(\Delta \lambda) y_{lmN}^{\text{down}*}(\Delta \lambda) \right\} \right] \\ &= 2 \left( \left\langle \frac{dE}{dt} \right\rangle_t \langle V_{tr} \rangle - \left\langle \frac{dL}{dt} \right\rangle_t \langle V_{\phi r} \rangle \right) \\ &\quad + \frac{q^2 \Upsilon_r}{32\pi^2} \sum_{\#} \frac{\omega_{mN}}{|\omega_{mN}|} \left\{ \frac{1}{|\alpha_{\omega_{mN}lm}|^2} Z_{lmN}^{\text{out}}(\Delta \lambda) y_{lmN}^{\text{out}*}(\Delta \lambda) + \frac{\omega_{mN} p_{m\omega_{mN}}}{|\omega_{mN} p_{m\omega_{mN}}|} \frac{|\tau_{\omega_{mN}lm}|^2}{|\beta_{\omega_{mN}lm}|^2} Z_{lmN}^{\text{down}}(\Delta \lambda) y_{lmN}^{\text{down}*}(\Delta \lambda) \right\} \end{aligned} \quad (74)$$

with  $\sum_{\#} := \sum_{l=0}^{+\infty} \sum_{m=-l}^l \sum_{N=-\infty}^{+\infty}$  and

$$y_{lmN}^b(\Delta \lambda) := \frac{1}{i \Upsilon_r} \frac{d}{d(\Delta \lambda)} Z_{lmN}^b(\Delta \lambda) = \frac{2\pi}{\Upsilon_t} \sum_{(n_r, n_\theta) \in \hat{N}} n_r \tilde{g}_{lmn_r n_\theta}^b e^{i n_r \Upsilon_r \Delta \lambda}. \quad (75)$$

In the second equality we used the expressions of  $\langle dE/dt \rangle_t$  and  $\langle dL/dt \rangle_t$  [12],

$$\left\langle \frac{d}{dt} \left\{ \begin{array}{c} E \\ L \end{array} \right\} \right\rangle_t = -\frac{q^2}{64\pi^2} \sum_{\#} \left\{ \begin{array}{c} \omega_{mN} \\ m \end{array} \right\} \frac{\omega_{mN}}{|\omega_{mN}|} \left[ \frac{|Z_{lmN}^{\text{out}}(\Delta \lambda)|^2}{|\alpha_{\omega_{mN}lm}|^2} + \frac{\omega_{mN} p_{m\omega_{mN}}}{|\omega_{mN} p_{m\omega_{mN}}|} \frac{|\tau_{\omega_{mN}lm}|^2}{|\beta_{\omega_{mN}lm}|^2} |Z_{lmN}^{\text{down}}(\Delta \lambda)|^2 \right]. \quad (76)$$

The contribution from the radiative field in Eq. (74) is nothing but the scalar analogue of  $\langle dQ/dt \rangle_t^{(\text{rad})}$  obtained by Flanagan *et al.* [17]. Notice that the above expression for the radiative part is also valid for the off-resonance case once we interpret that the summation over  $N$  means the summation over independent frequencies in this case.

### C. Practical formula of $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$ : the regularized symmetric part

To obtain a more practical expression for  $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$ , we rewrite the potential functions of the symmetric and  $S$ -parts given in Eq. (63), respectively. The symmetric part can be treated in a similar manner to the radiative part discussed in Sec. III B, though we cannot completely separate the averages over  $\lambda$  and  $\lambda'$  at the level of each mode. Substituting Eq. (37) (with Eq. (38)) into the potential function of the symmetric part (63), we obtain

$$\begin{aligned} \Psi^{(\text{sym})}(\Delta \lambda; \epsilon, \zeta) &= \frac{1}{16\pi} \int_{-\infty}^{\infty} d\omega \sum_{lm} e^{-i\omega \epsilon \cos \zeta + i m \epsilon (\Omega_\phi \cos \zeta - \Omega \sin \zeta)} \\ &\quad \times \int_{-\infty}^{\infty} d\lambda' \left\langle e^{-i(\lambda - \lambda')(\omega \Upsilon_t - m \Upsilon_\phi)} \tilde{g}_{\omega lm}^{(\text{sym})} [\Delta \bar{z}(\lambda, \lambda - \Delta \lambda), \Delta \bar{z}(\lambda', \lambda' - \Delta \lambda)] \right\rangle_\lambda, \end{aligned} \quad (77)$$

where

$$\tilde{g}_{\omega lm}^{(\text{sym})}(x, x') := \Sigma(x)\Sigma(x')g_{\omega lm}^{(\text{sym})}(x, x').$$

Since  $\tilde{g}_{\omega lm}^{(\text{sym})}[\Delta\bar{z}(\lambda, \lambda - \Delta\lambda), \Delta\bar{z}(\lambda', \lambda' - \Delta\lambda)]$  is a periodic function with period  $\Lambda$ , we find that the  $\lambda'$ -integral in Eq. (77) produces  $\delta(\omega - \omega_{mN})$ , and therefore we can discretized the Fourier integral as

$$\Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta) = \frac{1}{8\Upsilon_t} \sum_{\#} e^{-i\Omega(\epsilon_1 N + \epsilon_2 m)} \frac{1}{\Lambda} \int_0^\Lambda d\lambda' \left\langle e^{-iN\Upsilon(\lambda - \lambda')} \tilde{g}_{\omega_{mN}lm}^{(\text{sym})}[\Delta\bar{z}(\lambda, \lambda - \Delta\lambda), \Delta\bar{z}(\lambda', \lambda' - \Delta\lambda)] \right\rangle_\lambda, \quad (78)$$

where,  $\epsilon_1 = \epsilon \cos \zeta$ , and  $\epsilon_2 = \epsilon \sin \zeta$ . In the above equation, the long-time average over  $\lambda$  can be replaced by an average over one period because the part in the angle brackets is periodic with period  $\Lambda$ . For later convenience in subtracting the the potential function of the  $S$ -part,  $\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$ , from  $\Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta)$ , we rewrite Eq. (78) by rearranging the order of the summation with respect to  $l$  and  $m$  as

$$\Psi^{(\text{sym})}(\Delta\lambda; \epsilon, \zeta) = \sum_{N=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{-i\Omega(\epsilon_1 N + \epsilon_2 m)} \Psi_{mN}^{(\text{sym})}(\Delta\lambda) \quad (79)$$

with

$$\Psi_{mN}^{(\text{sym})}(\Delta\lambda) := \frac{1}{8\Upsilon_t} \sum_{l=|m|}^{\infty} \frac{1}{\Lambda^2} \int_0^\Lambda d\lambda \int_0^\Lambda d\lambda' e^{-iN\Upsilon(\lambda - \lambda')} \tilde{g}_{\omega_{mN}lm}^{(\text{sym})}[\Delta\bar{z}(\lambda, \lambda - \Delta\lambda), \Delta\bar{z}(\lambda', \lambda' - \Delta\lambda)]. \quad (80)$$

We interpret Eqs. (79) and (80) as the Fourier expansion in the  $(\epsilon_1, \epsilon_2)$ -space and the Fourier coefficients, respectively. The point is that each  $(m, N)$  Fourier mode is finite, which can be calculated numerically.

Next, we derive  $\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$ . Using half the squared geodesic distance between  $x$  and  $x'$ ,  $\sigma(x, x')$ , the  $S$ -part Green function defined in the local convex neighborhood of the particle's position is written as

$$G^{(S)}(x, x') = -\frac{1}{8\pi} [U(x, x')\delta(\sigma(x, x')) - V(x, x')\theta(\sigma(x, x'))] \quad (81)$$

with

$$U(x, x') = 1 + O(|x - x'|^3/L^3), \quad (82)$$

where  $L$  is the scale of curvature of the background spacetime, and  $V(x, x')$  is a regular function in the coincidence limit  $x \rightarrow x'$ . (See, e.g., Ref. [2] for more concrete definitions of these quantities.) The integrand containing  $V(x, x')$  contributes as the finite value to  $\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$  since  $\lambda'$  integral has its support only in a short interval around  $\lambda' \approx \lambda$ . In the coincidence limit, this interval shrinks to zero, and thus this contribution also vanishes. Then, we need to consider only the term including  $U(x, x')$ . The argument of the  $\delta$ -function is expanded, abbreviating the argument  $\Delta\lambda$ , as

$$\begin{aligned} \sigma[z_+(\lambda), z_-(\lambda')] &= \frac{1}{2} g_{\mu\nu}[z(\bar{\lambda})] \{z^\mu(\lambda) - z^\mu(\lambda') + \epsilon\xi^\mu\} \{z^\nu(\lambda) - z^\nu(\lambda') + \epsilon\xi^\nu\} + O(\epsilon^4) \\ &= \frac{1}{2} g_{\mu\nu}[z(\bar{\lambda})] \{\Sigma[z(\bar{\lambda})]u^\mu(\bar{\lambda})\delta\lambda + \epsilon\xi^\mu\} \{\Sigma[z(\bar{\lambda})]u^\nu(\bar{\lambda})\delta\lambda + \epsilon\xi^\nu\} + O(\epsilon^4), \end{aligned} \quad (83)$$

where we introduced new variables

$$\delta\lambda := (\lambda - \lambda'), \quad \bar{\lambda} := \frac{1}{2}(\lambda + \lambda'), \quad (84)$$

and assumed  $\delta\lambda = O(\epsilon)$  since the solution of  $\sigma = 0$  for  $\delta\lambda$  is so. We stress that  $O(\epsilon^3)$  term is absent since  $\sigma$  is unchanged under the transformation  $\epsilon \rightarrow -\epsilon$  and  $\delta\lambda \rightarrow -\delta\lambda$ . With the aid of the above expansion for  $\sigma$ , expanding both  $\Sigma[z(\lambda)]$  and  $\Sigma[z(\lambda')]$  in terms of  $\epsilon$  as well, we eventually obtain

$$\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta) = -\frac{\psi(\zeta)}{\epsilon} + O(\epsilon) \quad (85)$$

with

$$\psi(\zeta) := \frac{1}{4\pi\Lambda} \int_0^\Lambda d\bar{\lambda} \frac{\Sigma(x)}{\sqrt{(g_{\mu\nu} + u_\mu u_\nu)\xi^\mu \xi^\nu}} \Big|_{x=z(\bar{\lambda})}, \quad (86)$$

where we replaced the long-time average over  $\bar{\lambda}$  by the average over one period because the integrand is periodic with period  $\Lambda$ . Note that  $\psi(\zeta)$  depends on  $\zeta$  through  $\xi^\mu$ . This is the only  $S$ -part contribution that remains in the limit  $\epsilon \rightarrow 0$ . Since  $(g_{\mu\nu} + u_\mu u_\nu)$  is a spacelike projection operator, the expression inside the square root is positive semi-definite.

To subtract the  $S$ -part of the potential function from the one of symmetric part mode by mode, we calculate the  $(m, N)$ -modes of  $\Psi^{(S)}(\Delta\lambda; \epsilon, \zeta)$  as

$$\begin{aligned}
\Psi_{mN}^{(S)}(\Delta\lambda) &= \frac{\Omega^2}{4\pi^2} \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{i\Omega(\epsilon_1 N + \epsilon_2 m)} \Psi^{(S)}(\Delta\lambda; \epsilon, \zeta) \\
&= -\frac{\Omega^2}{4\pi^2} \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{i\Omega(\epsilon_1 N + \epsilon_2 m)} \left( \frac{\psi(\zeta)}{\sqrt{\epsilon_1^2 + \epsilon_2^2}} \right) \\
&= -\frac{\Omega^2}{4\pi^2} \left[ \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\zeta}{\cos \zeta} \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 e^{i\epsilon_1 \Omega(N + m \tan \zeta)} \psi(\zeta) - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{d\zeta}{\sin \zeta} \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{i\epsilon_2 \Omega(N \cot \zeta + m)} \psi(\zeta) \right] \\
&= \frac{\Omega}{2\pi^2} \left[ (-1)^m \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\zeta \frac{\sin(\pi N \cot \zeta) \psi(\zeta)}{N \cos \zeta + m \sin \zeta} - (-1)^N \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\zeta \frac{\sin(\pi m \tan \zeta) \psi(\zeta)}{N \cos \zeta + m \sin \zeta} \right]. \tag{87}
\end{aligned}$$

This expression is a double integral with respect to  $\bar{\lambda}$  and  $\zeta$ , and it is finite. Since  $\psi(\zeta)$  is independent of  $m$  and  $N$ , once we obtain  $\psi(\zeta)$  as a function of  $\zeta$  by performing the  $\bar{\lambda}$ -integral, the remaining integral over  $\zeta$  for each pair of  $m$  and  $N$  is just a single integral.

We stress that subtracting Eq. (87) from Eq. (80) is a variant of the so-called mode sum regularization [1, 27] developed in the context of self-forces, but our proposal is suitable for the decomposition in terms of *the spheroidal harmonics*. We do not have to reexpand the perturbation using the spherical harmonics, which is usually utilized. After subtracting this  $S$ -part contribution from  $\Psi_{mN}^{(\text{sym})}(\Delta\lambda)$  mode by mode, one should be able to take the summations over  $m$  and  $N$  without any divergence. The convergence of these summations will be even accelerated by fitting the asymptotic form as usual. We will report the result of explicit convergence test of  $\Psi_{mN}^{(\text{sym})}(\Delta\lambda) - \Psi_{mN}^{(S)}(\Delta\lambda)$  in our succeeding publication.

#### IV. SUMMARY AND FUTURE DIRECTIONS

In this paper, we have derived the long-time averaged rate of change of the Carter constant,  $\langle dQ/dt \rangle_t$ , for a self-interacting scalar charge within the adiabatic regime, which is also applicable to a resonant inspiral. The essential point is that, by contrast to off-resonance orbits, a resonance orbit depends on the offset phase,  $\Delta\lambda$ , which is the difference of the time when the  $r$ - and  $\theta$ -oscillations reach their minimum values. While this offset phase is identically zero for a off-resonance orbit, it takes a non-zero value for a resonance orbit, and also evolves slowly compared to the time scale of the orbital motion. Therefore, we need to care about the evolution of not only the three constants of motion but also  $\Delta\lambda$ . The existence of a non-zero  $\Delta\lambda$  requires a direct computation of the self-forces acting on the charged particle even for calculating the averaged rate of change of the Carter constant. This means that  $\langle dQ/dt \rangle_t$  for a resonance orbit is no longer determined solely by the radiative field  $\Phi^{(\text{rad})}(x)$  alone, and that we need to consider the contribution from the symmetric field,  $\Phi^{(\text{sym})}(x)$  as well.

As for the radiative part  $\langle dQ/dt \rangle_t^{(\text{rad})}$ , we derived a practical expression in a similar manner to an off-resonance case with slight modification, and found that our result is equivalent to the scalar analogue of the long-time averaged rate for the gravitational case shown in Ref. [17]. On the other hand, the symmetric part is more complicated than the radiative one because of the singular behaviour of the symmetric field at the location of the particle. We proposed a method of subtracting the singular behaviour, so-called  $S$ -part, from the symmetric part via the point splitting regularization in the Killing directions. The key point is that it is compatible with the Teukolsky formalism, which makes the calculation much simpler than the direct self-force calculation. While there is an expectation that  $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$  might eventually turn to be suppressed, which is suggested by post-Newtonian analysis [28], this conjecture should be tested in the fully general relativistic analysis. Since the numerical code that calculates the long-time average of the constants of motion for a generic off-resonance orbit around the Kerr black hole is well developed [29, 30], there is in principle no obstacle to numerically test the conjecture. The code is now under development and the result will be reported in our succeeding publication.

To avoid unnecessary confusion over various works, we comment on the following two points related to our current works. The first point is the difference between the long-time average and the phase space average for a resonance orbit, discussed in Refs. [6, 13]. Different from the long-time average, the phase space average for a resonance orbit means that the averaging is further performed with respect to the slowly evolving variable  $\Delta\lambda$  that is an important control variable of the resonance orbit. The phase space average eliminates all dependence of  $\Delta\lambda$  both in  $\langle dQ/dt \rangle_t^{(\text{rad})}$  as well as  $\langle dQ/dt \rangle_t^{(\text{sym}-S)}$ , thus it is likely to miss the important contribution to the accumulated orbital phase correction that would exceed  $O(1)$  radians. (See discussion in

Appendix A.) For this reason, we believe that only the long-time averaged rate of the change of constants of motion can be a good candidate to obtain the adiabatic evolution of a resonant inspiral with phase accuracy better than  $O(1)$ .

The other point is that our new method does not reduce the importance of developing the direct self-force calculation that is recently under a rapid progress. We think that these two methods are complimentary. Relatively high precision will be required in evaluating the effect that may cause an orbital phase error enhanced by an inverse power of the mass ratio. Numerical evaluation of such quantities might be a tough problem if we cannot take advantage of the Teukolsky formalism that reduces the problem into the one to solve one-dimensional radial functions. Even if achieving high accuracy numerically is not so difficult, our method will give reference numbers which are useful to check the accuracy of the results obtained by direct self-force calculations.

Before closing this paper, we would like to mention how to extend our method to the case of the gravitational radiation reaction. Since our method is based on the mode sum regularization in which each mode contribution is gauge invariant, we do not have to care about the gauge adjustment between the symmetric part and the  $S$ -part. Even for the gravitational case, the contribution from the  $S$ -part evaluated in the Lorenz gauge is almost identical to the scalar case discussed in this paper. To calculate the contribution from the symmetric part, we can use the standard Teukolsky formalism with the aid of the established method of reconstructing the metric perturbation from the master variables [31–34]. The metric perturbation reconstructed in this manner is given in the so-called radiation gauge, in which a singularity runs from the particle's trajectory in the radial null direction. However, almost all regularized trajectories shifted in the Killing directions do not cross this singularity. Therefore we expect that the presence of this radial singularity will not cause any trouble. Although we need further study for practical implementation, we believe that our method can be extended to the gravitational case as briefly described above,

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### Appendix A: The possible pattern for the long-term orbital evolution crossing the resonance

The final goal of this project is understanding the long-time orbital evolution crossing the resonance in the gravitational case. Toward this ambitious goal, this appendix is dedicated to quantitatively discuss how we can understand the long-term evolution of the resonant inspiral, as a future application of our formalism. In this appendix, we consider the gravitational case: a point particle with the rest mass  $\mu$  moves along the quasi-resonant orbits in the Kerr black hole with the rest mass  $M$  and the azimuthal angular momentum  $aM$ . To apply the following discussion to the scalar model, we just need to replace  $\mu$  to  $(q^2/\mu)$  in the results, where  $q$  is the scalar charge of a point scalar particle.

As was discussed in Ref. [5, 9], as far as we consider the first order self-force in the osculating orbit approximation, the long-term orbital evolution is basically determined by tracing the evolution of slowly changing variables, *i.e.*, the constants of motion  $I^i$ , and the orbital frequencies as their functions,  $\Upsilon_a(I^i)$ . We only need to know the non-oscillating part of the rates of change of  $I^i$  and  $\Upsilon_a(I^i)$  for a generic orbit as the corrections due to the self-force. In particular, we only need to know the former if it is allowed to neglect  $O(1)$  error in the orbital phase. This observation is essentially equivalent to the more systematic two-time scale analysis given in Ref. [6].

Near the resonance, however, the adiabatic evolution of the orbit cannot be described by tracing the constants of motion  $I^i$  because the self-force also depends on the offset phase  $\Delta\lambda$ , the relative phase between  $r$ - and  $\theta$ -oscillations, as we have discussed in the main text. Neglecting the part that oscillates in the orbital time scale, the time derivatives of  $I^i$  are determined as functions of  $I^i$  and  $\Delta\lambda$ , which we denote by  $\dot{I}^i(I^i, \Delta\lambda)$ . At the same time, however,  $\Delta\lambda$  also evolves unless the exact resonance condition is satisfied, and the resonance orbit effectively comes back to the off-resonance orbit when the offset phase starts to change rapidly. We examine possible patterns of evolution of  $\{I^i, \Delta\lambda\}$  in this section.

Suppose that the resonance condition is only approximately satisfied, but its deviation is small. In this situation,  $\dot{I}^i$  may still depend on  $\Delta\lambda$ . Moreover, we are also allowed to replace  $I^i$  in  $\dot{I}^i(I^i, \Delta\lambda)$  by the fixed value at the exact resonance point,



which we denote  $I_{\text{res}}^i$ . This can be understood in the following manner. The typical time scale for crossing the resonance point is estimated as  $\tau_{\text{res}} \sim O(\Omega_\phi \sqrt{M/\mu})$ . During the resonance crossing, the net change of the constants of motion from  $I_{\text{res}}^i$  is estimated as  $\Delta I^i \sim \dot{I}^i \tau_{\text{res}} \sim O(\Omega_\phi \sqrt{\mu/M})$  [5, 6]. This means that the correction to  $I_{\text{res}}^i$  is always suppressed by the factor of  $O(\sqrt{\mu/M})$  in resonance crossing. Based on this observation, we can estimate the error in  $\dot{I}^i$  that associates with the replacement of its argument from  $I^i$  to  $I_{\text{res}}^i$  as

$$\frac{d}{dt} (\log I^i(I^i, \Delta\lambda)) = \frac{d}{dt} (\log I_{\text{res}}^i(I^i, \Delta\lambda)) + O\left(\Omega_\phi \left(\frac{\mu}{M}\right)^{3/2}\right). \quad (\text{A1})$$

After passing the resonance, this error term causes the shifts of the constants of motion, which will be evaluated as  $\Delta I_{\text{err}}^i/I^i \sim O(\mu/M)$ . However, the net phase error due to these shifts, which is simply given by  $\Delta I_{\text{err}}^i$  times  $\tau_{\text{rad}}$  is at most  $O(1)$ . This phase error is small compared with what we are interested in here. Thus, we conclude that the evolution of the constants of motion during the resonance crossing is given by  $\dot{I}^i(I_{\text{res}}^i, \Delta\lambda)$ .

Next, we discuss the evolution of the offset phase,  $\Delta\lambda$ . In the situation considered here, there is no longer exact common resonance period  $\Lambda$ . Instead, we introduce the approximate common period  $\tilde{\Lambda}$  by

$$\tilde{\Lambda} := j_\theta \Lambda_\theta = \frac{2\pi j_\theta}{\Upsilon_\theta}. \quad (\text{A2})$$

During the period  $\tilde{\Lambda}$ , the offset phase is shifted by

$$\delta(\Delta\lambda) = \tilde{\Lambda} - j_r \Lambda_r = \frac{\Lambda_r \Lambda_\theta}{2\pi} \Delta\Upsilon, \quad (\text{A3})$$

where  $\Delta\Upsilon := j_\theta \Upsilon_r - j_r \Upsilon_\theta$ . When  $\Delta\Upsilon = 0$ , we have the exact resonance. In the adiabatic regime, we are also allowed to replace the time derivative of  $\Delta\lambda$  to the averaged value over one period  $\tilde{\Lambda}$ . As a result, the evolution of  $\Delta\lambda$  is described as

$$\left\langle \frac{d\Delta\lambda}{dt} \right\rangle_t = \Upsilon_t^{-1} \frac{\delta(\Delta\lambda)}{\tilde{\Lambda}} = \frac{\Delta\Upsilon}{j_\theta \Upsilon_t \Upsilon_r}. \quad (\text{A4})$$

Now, we investigate the evolution of  $\Delta\lambda$  around a resonance point in more detail. For this purpose, focusing on the time derivative of  $\langle d\Delta\lambda/dt \rangle_t$ , we consider the two cases, separately, depending on whether or not there exists  $\Delta\lambda_c$  such that

$$\frac{d}{dt} \left\langle \frac{d\Delta\lambda}{dt} \right\rangle_t = \frac{\partial}{\partial I^j} \left( \frac{\Delta\Upsilon}{j_\theta \Upsilon_t \Upsilon_r} \right) \dot{I}^j(I^i, \Delta\lambda_c) = 0. \quad (\text{A5})$$

(See also Ref. [35] for the classification of resonance orbits.) The latter case has been already discussed in Ref. [5] and is essentially the same as the transient resonance analysed in Refs. [13, 17]. In this case,  $\Delta\lambda$  varies of  $O(1)$  on the time scale of  $O(\Omega_\phi/\sqrt{\mu})$  around  $\Delta\Upsilon = 0$ , during which  $I^i$  evolve by  $O(\sqrt{\mu})$  relatively. The effect of the resonance alters the evolution of  $I^i$  which leads to the phase corrections of  $O(\sqrt{\mu}\Omega_\phi\Delta T)$ , where  $\Delta T$  is the time interval between the resonance and the plunge. As typically  $\Delta T$  scales like  $(\propto 1/\mu)$ , the phase corrections due to this effect can be significantly large  $(\propto 1/\sqrt{\mu})$  if the inspiral in the extreme mass ratio regime,  $M \gg \mu$ .

A more interesting situation may happen if  $\Delta\lambda_c$  exists. This case corresponds to the so-called sustained resonance [13]. When  $\Delta\lambda_c$  is sufficiently close to  $\Delta\lambda_c$ , we obtain, with Taylor expansion around  $\Delta\lambda_c$ ,

$$\frac{d}{dt} \left\langle \frac{d\Delta\lambda}{dt} \right\rangle_t = -\varpi^2(I^i)(\Delta\lambda - \Delta\lambda_c(I^i)), \quad (\text{A6})$$

where

$$\varpi^2(I^i) := -\frac{\partial}{\partial I^j} \left( \frac{\Delta\Upsilon}{j_\theta \Upsilon_t \Upsilon_r} \right) \frac{\partial \dot{I}^j}{\partial \Delta\lambda}(I^i, \Delta\lambda_c(I^i)).$$

Since the time scale  $1/|\varpi|$  scales like  $(\propto 1/\sqrt{\mu})$ , it is likely to be much shorter than the radiation reaction time scale, which scales like  $(\propto 1/\mu)$ . If  $\varpi^2 > 0$ , the above system behaves like a harmonic oscillator with its potential minimum varying adiabatically. In this case,  $\Delta\lambda$  oscillates around  $\Delta\lambda_c$  with the amplitude  $\propto 1/\sqrt{\varpi}$ . This evolution may last as long as the condition  $\varpi^2 > 0$  and the existence of  $\Delta\lambda_c$  are maintained. Therefore, in principle, the resonance may last for the radiation reaction time scale. This time scale is much longer than we naively expect when the possible presence of  $\Delta\lambda_c$  is neglected. If  $\varpi^2 < 0$ , the value of  $\Delta\lambda$  close to  $\Delta\lambda_c$  is disfavored. The time scale for the variation of  $\Delta\lambda$  is again  $\propto 1/\sqrt{\mu}$  as in the case of the absence of  $\Delta\lambda_c$ . Hence, the corrections to the phase evolution are also of the same order as before. Although the post-Newtonian analysis [13] and the gravitational wave fluxes from a resonance orbit [17] suggest that the sustained resonance may be absent in inspirals around the Kerr black hole, or exist only in very restricted orbital parameter regions even if it is achieved, the existence of the sustained resonance in inspirals into the Kerr black hole is still an open question.



### Appendix B: Derivation of Eq. (59)

First, to show Eqs. (57) and (58), we clarify the properties of  $z^\mu(\lambda)$  and  $G^{(\text{sym}-S)}(x, x')$ , respectively. For resonance orbits,  $r(\lambda)$  and  $\theta(\lambda)$  are periodic with period  $\Lambda (= j_r \Lambda_r = j_\theta \Lambda_\theta)$  as shown in Eq. (7), while  $t(\lambda)$  and  $\phi(\lambda)$  have secular evolution terms proportional to  $\lambda$  in addition to oscillatory parts with period  $\Lambda$  as Eqs. (12) and (14). Taking account of these facts, we find that, under the transformation  $\lambda \rightarrow \lambda + \Lambda$ ,  $z^\mu(\lambda)$  transforms as

$$z^\mu(\lambda + \Lambda) = z^\mu(\lambda) + \left[ \Upsilon_t \xi_{(t)}^\mu + \Upsilon_\phi \xi_{(\phi)}^\mu \right] \Lambda = z^\mu(\lambda) + \Upsilon_t \Lambda \xi^\mu(0), \quad (\text{B1})$$

where  $\xi^\mu(\zeta)$  is defined in Eq. (61). This means that the trajectory is displaced in a Killing direction.

As for  $G^{(\text{sym}-S)}(x, x')$ , we find two properties. Due to the stationarity and axisymmetry of the Kerr spacetime, the Green functions are invariant under the displacement along an arbitrary Killing vector field:

$$G(x + A\xi(\zeta), x' + A\xi(\zeta)) = G(x, x'), \quad (\text{B2})$$

where  $A$  is an arbitrary constant. From Eqs. (B1) and (B2), we can derive Eq. (57) as

$$\begin{aligned} G(z(\lambda + \Lambda), z(\lambda' + \Lambda)) &= G(z(\lambda) + \Upsilon_t \Lambda \xi(0), z(\lambda') + \Upsilon_t \Lambda \xi(0)) \\ &= G(z(\lambda), z(\lambda')). \end{aligned}$$

The relation (B2) simply tells that the Green functions depend on  $t$  and  $t'$  through  $t - t'$ , and on  $\phi$  and  $\phi'$  through  $\phi - \phi'$ . Based on this understanding, we find that Eq. (58) holds.

Next, we show that the first term on the right hand side of Eq. (55) vanishes by using Eqs. (56), (57), and (58). We focus only on the  $t$ -derivative part, but one can show that the  $\phi$ -derivative part vanishes in the same way. By using the property (57), we can replace the long-time average over  $\lambda$  to the average over one period  $\Lambda$  as

$$\begin{aligned} &\int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] \left\langle [\Sigma(x) \partial_t G(x, z(\lambda'))]_{x=z(\lambda)} \right\rangle_\lambda \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N\Lambda} \sum_{n=-N}^{N-1} \int_{n\Lambda}^{(n+1)\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N\Lambda} \sum_{n=-N}^{N-1} \int_0^\Lambda d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \frac{1}{\Lambda} \int_0^\Lambda d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')}. \end{aligned} \quad (\text{B3})$$

In the third equality, we transformed  $(\lambda, \lambda')$  to  $(\lambda - n\Lambda, \lambda' - n\Lambda)$ , and used Eq. (57) and  $\Sigma[z(\lambda - n\Lambda)] = \Sigma[z(\lambda)]$  (note that  $\Sigma(x)$  is a function of  $r$  and  $\theta$ ). We also rewrite the integrals over  $\lambda$  and  $\lambda'$  as

$$\begin{aligned} &\frac{1}{\Lambda} \int_0^\Lambda d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \frac{1}{\Lambda} \sum_{n=-\infty}^{\infty} \int_0^\Lambda d\lambda \int_{n\Lambda}^{(n+1)\Lambda} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \frac{1}{\Lambda} \sum_{n=-\infty}^{\infty} \int_{-n\Lambda}^{-(n-1)\Lambda} d\lambda \int_0^\Lambda d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')} \\ &= \frac{1}{\Lambda} \int_{-\infty}^{\infty} d\lambda \int_0^\Lambda d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] [\partial_t G(x, x')]_{x=z(\lambda), x'=z(\lambda')}. \end{aligned} \quad (\text{B4})$$

Further, for the Green function  $G^{(\text{sym}-S)}(x, x')$ , by exchanging the labels in the last line of Eq. (B4) as  $\lambda \leftrightarrow \lambda'$  and  $x \leftrightarrow x'$ , we

obtain

$$\begin{aligned}
& \frac{1}{\Lambda} \int_{-\infty}^{\infty} d\lambda \int_0^{\Lambda} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] \left[ \partial_t G^{(\text{sym}-S)}(x, x') \right]_{x=z(\lambda), x'=z(\lambda')} \\
&= \frac{1}{\Lambda} \int_0^{\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] \left[ \partial_t G^{(\text{sym}-S)}(x', x) \right]_{x=z(\lambda), x'=z(\lambda')} \\
&= \frac{1}{\Lambda} \int_0^{\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda)] \Sigma[z(\lambda')] \left[ -\partial_t G^{(\text{sym}-S)}(x, x') \right]_{x=z(\lambda), x'=z(\lambda')}, \tag{B5}
\end{aligned}$$

where we used Eqs. (56) and (58) in the last equality. Therefore we find

$$\int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] \left\langle \left[ \Sigma(x) \partial_t G^{(\text{sym}-S)}(x, z(\lambda')) \right]_{x=z(\lambda)} \right\rangle_{\lambda} = 0. \tag{B6}$$

As for the  $\Delta\lambda$ -derivative part, we transform

$$\begin{aligned}
& \int_{-\infty}^{\infty} d\lambda' \left[ \Sigma(x') \frac{d}{d(\Delta\lambda)} \left\langle \Sigma[z(\lambda)] G^{(\text{sym}-S)}(z(\lambda), x') \right\rangle_{\lambda} \right]_{x'=z(\lambda')} \\
&= \frac{1}{\Lambda} \int_0^{\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \left[ \Sigma(x) \frac{d}{d(\Delta\lambda)} \left\{ \Sigma[z(\lambda')] G^{(\text{sym}-S)}(x, z(\lambda')) \right\} \right]_{x=z(\lambda)}, \tag{B7}
\end{aligned}$$

where we replaced the long-time average with respect to  $\lambda$  with the average over one orbital period  $\Lambda$  as shown in Eq. (B3), and exchanged the labels as  $\lambda \leftrightarrow \lambda'$  and  $x \leftrightarrow x'$ , and the domains of integration with respect to  $\lambda$  and  $\lambda'$  in the same way as in Eq. (B4). Using Eq. (B7), we find

$$\begin{aligned}
& \int_{-\infty}^{\infty} d\lambda' \left[ \Sigma(x') \frac{d}{d(\Delta\lambda)} \left\langle \Sigma[z(\lambda)] G^{(\text{sym}-S)}(z(\lambda), x') \right\rangle_{\lambda} \right]_{x'=z(\lambda')} \\
&= \frac{1}{2\Lambda} \int_0^{\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \left\{ \left[ \Sigma(x') \frac{d}{d(\Delta\lambda)} \left\{ \Sigma[z(\lambda)] G^{(\text{sym}-S)}(z(\lambda), x') \right\} \right]_{x'=z(\lambda')} \right. \\
&\quad \left. + \left[ \Sigma(x) \frac{d}{d(\Delta\lambda)} \left\{ \Sigma[z(\lambda')] G^{(\text{sym}-S)}(x, z(\lambda')) \right\} \right]_{x=z(\lambda)} \right\} \\
&= \frac{1}{2\Lambda} \int_0^{\Lambda} d\lambda \int_{-\infty}^{\infty} d\lambda' \frac{d}{d(\Delta\lambda)} \left[ \Sigma[z(\lambda)] \Sigma[z(\lambda')] G^{(\text{sym}-S)}(z(\lambda), z(\lambda')) \right] \\
&= \frac{1}{2} \frac{d}{d(\Delta\lambda)} \int_{-\infty}^{\infty} d\lambda' \left\langle \Sigma[z(\lambda)] \Sigma[z(\lambda')] G^{(\text{sym}-S)}(z(\lambda), z(\lambda')) \right\rangle_{\lambda'}. \tag{B8}
\end{aligned}$$

Removing the  $t$ - and  $\phi$ -derivative parts from Eq. (55) and using Eq. (B8), we finally obtain Eq. (59).

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